Solutions

3.3: Higher-Order Linear, Homogeneous Equations with Constant Coefficients Real Roots

Theorem 1. (Distinct Real Roots)
If the roots $r_1, r_2, \ldots, r_n$ of the characteristic equation (2) are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

is a general solution to (1).

Exercise 1. Solve the initial value problem

$$y^{(3)} + 3y'' - 10y' = 0, \quad y(0) = 7, y'(0) = 0, y''(0) = 70.$$

$r^3 + 3r^2 - 10r = r(r^2 + 3r - 10) = r(r+5)(r-2) \Rightarrow r = 0, -5, 2$

$y = c_1 + c_2 e^{-5x} + c_3 e^{2x} \Rightarrow y(0) = 7 = c_1 + c_2 + c_3$

$y' = -5c_2 e^{-5x} + 2c_3 e^{2x} \Rightarrow y'(0) = 0 = -5c_2 + 2c_3$

$y'' = 25c_2 e^{-5x} + 4c_3 e^{2x} \Rightarrow y''(0) = 70 = 25c_2 + 4c_3$

$\Rightarrow \quad c_1 = 0, \; c_2 = 2, \; c_3 = 5$

$y = 2e^{-5x} + 5e^{2x}$

Theorem 2. (Repeated Roots)
If the characteristic equation (2) has a repeated real root r of multiplicity k, then the part of a general solution of the differential equation (1) corresponding to r is of the form

$$(c_1 + c_2 x + \cdots + c_{k-1} x^{k-2} + c^k x^{k-1}) e^{rx}.$$

Exercise 2. Find a general solution to the equation

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0.$$

$$9r^5 - 6r^4 + r^3 = 0$$

$$0 = r^3(9r^2 - 6r + 1) = r^3(9r^2 - 3r - 3r + 1) = r^3(3r(3r-1) - (3r-1))$$

$$= r^3((3r-1)^2)$$

$$r = 0, 0, 0, \tfrac{1}{3}, \tfrac{1}{3}$$

$$y = C_1 + C_2 x + C_3 x^2 + C_4 e^{\frac{1}{3}x} + C_5 x e^{\frac{1}{3}x}$$